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A graph theoretical approach to *cis/trans* isomerism

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Abstract: A simple graph-theory-based model is presented, by means of which it is possible to express the energy difference between geometrically non-equivalent forms of a conjugated polyene. This is achieved by modifying the adjacency matrix of the molecular graph, and including into it information on *cis/trans* constellations. The total π -electron energy thus calculated is in excellent agreement with the enthalpies of the underlying isomers and conformers.

Keywords: *cis/trans* isomerism; graph theory; molecular graph; total π -electron energy.

INTRODUCTION

One of the most unpleasant shortcomings of the modeling of molecular properties by means of the graph theory^{1–6} is the fact that a molecular graph does not contain information on the actual geometry of the underlying molecule. Consequently, the various graph-based molecular structure descriptors (topological indices) used in contemporary mathematical chemistry⁶ cannot distinguish between *cis/trans* isomers. Numerous attempts have been made to overcome this difficulty. Most of these construct 3-dimensional (3D) analogues of topological indices, calculated by using the actual (experimentally or theoretically determined) molecular geometry.^{7–10} This approach is often successful, but it abandons the usage of the graph theory.

In order to distinguish between *cis/trans* isomers while remaining within the domain of the chemical graph theory, Pogliani constructed special, *ad hoc* defined, types of molecular graphs.^{11,12} A much more direct approach was proposed by one of the present authors,¹³ utilizing a complex number-based adjacency matrix $\mathbf{A}(G)$ of the form:

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$$\mathbf{A}(G) = \mathbf{A}_1(G) + i\mathbf{A}_2(G) \quad (1)$$

where $i = \sqrt{-1}$ is the imaginary unit. In formula (1), $\mathbf{A}_1(G)$ is the ordinary adjacency matrix defined in the usual manner:^{1,3,4} If G is a (molecular) graph with vertices $v_1, v_2, \dots, v_n$, then the (i, j) -entry of $\mathbf{A}_1(G)$ is equal to 1 if the vertices v_i and v_j are adjacent, and is zero otherwise.

The imaginary part of $\mathbf{A}_1(G)$ is defined as follows.¹³ Let the topological distance (= the number of edges in a shortest path) between the vertices v_i and v_j of the graph G be denoted by $d(v_i, v_j | G)$. Then:

$$\begin{aligned} [\mathbf{A}_2(G)]_{ij} &= +1 \quad \text{if } d(v_i, v_j | G) = 3 \quad \& \quad v_i, v_j \text{ are in the } \textit{cis} \text{ constellation} \\ &= -1 \quad \text{if } d(v_i, v_j | G) = 3 \quad \& \quad v_i, v_j \text{ are in the } \textit{trans} \text{ constellation} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The advantage of a topological index, TI , calculated by means of Eq. (1) is that the contribution coming from the molecular topology (the real part of TI) is easily distinguished from the contribution coming from the molecular geometry (the imaginary part of TI). It was demonstrated¹³ that such an approach worked well in the case of standard graph-based structure descriptors, such as the connectivity index and the Wiener index (see recent reviews^{14,15}). However, as shown below, because the matrix $\mathbf{A}_1(G)$ is not Hermitian and, therefore, its eigenvalues are not real-valued, this approach is not applicable in the case of the total π -electron energy and the thermodynamic properties of polyenes.

In order to be able to check our models, first, data on the energetics of the *cis/trans* isomers of conjugated polyenes had to be acquired.

QUANTUM-MECHANICAL CALCULATIONS

In this work, attention was focused on isomers and (planar) conformers of 1,3,5-hexatriene and 2-vinylbutadiene, shown in Fig. 1. The enthalpy of formation of the 9 conjugated triene species of formula C_6H_8 were calculated using Gaussian software.

First, the geometries were optimized, and then – using the B3LYP/6--31G(d, p) basis set – the enthalpies were calculated. The obtained values are given in Table I.

RESULTS FOR THE MODEL BASED ON EQ. (1)

The HMO total π -electron energy, E_π is a much studied, graph-based characteristic of unsaturated conjugated molecules, see the books,^{2,4,16} surveys,^{17,18} recent papers,^{19–21} and references cited therein. Detailed examinations^{4,22,23} revealed that in the case of strain-free conjugated systems, there is a very good linear correlation between E_π and thermodynamic stability, in particular the enthalpy of formation.

Denote by $\lambda_1, \lambda_2, \dots, \lambda_n$ the eigenvalues^{2,4} of the (ordinary) adjacency matrix $\mathbf{A}_1(G)$ of the molecular graph G , and recall that these eigenvalues are always

real-valued. Then for alternant conjugated hydrocarbons, to which the polyenes studied in this work belong, the HMO total π -electron energy is given by:^{2,4,16}

$$E_{\pi} = \sum_{k=1}^n |\lambda_k| \quad (2)$$

Recall that $|x|$ denotes the absolute value of the number x . If x is real-valued, then $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x \leq 0$.

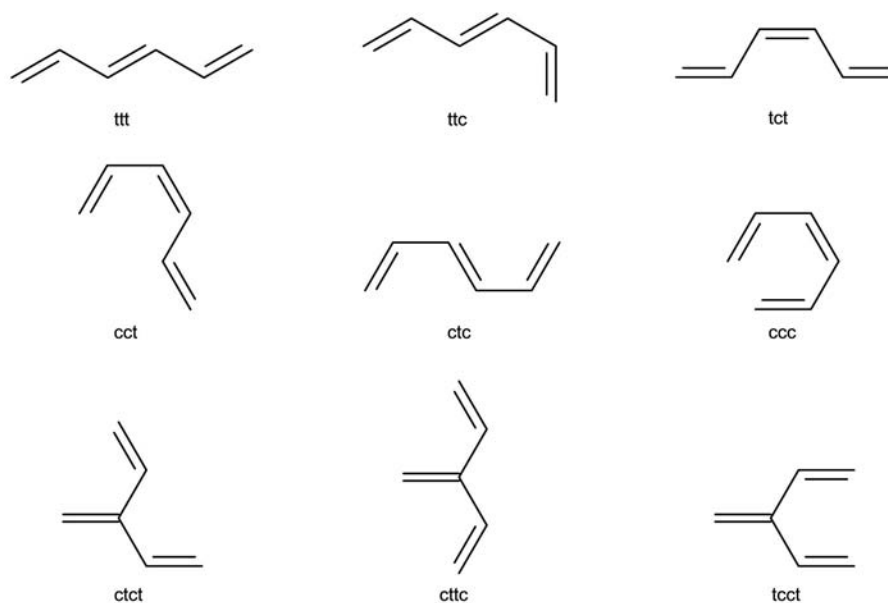


Fig. 1. Isomers and conformers of the conjugated hexatriene and vinylbutadiene, C_6H_8 , examined in this work.

TABLE I. Enthalpies of formation (in kJ mol^{-1}) for the C_6H_8 -species depicted in Fig. 1. The data in the third column were obtained from those in the second column, by subtracting the smallest value (the enthalpy of the *ttt*-species) and by multiplying by 1000

Isomer/conformer	$\Delta_f H$	$\Delta_f H_{\text{nor}}$
<i>ttt</i>	-233.284653	0.000
<i>ttc</i>	-233.279004	5.649
<i>tct</i>	-233.281451	3.202
<i>ctc</i>	-233.272950	11.703
<i>cct</i>	-233.275413	9.240
<i>ccc</i>	-233.265688	18.965
<i>ctct</i>	-233.274059	10.584
<i>cttc</i>	-233.270447	14.206
<i>tcct</i>	-233.272929	11.724

If an analogous expression for the modified adjacency matrix $\mathbf{A}(G)$, defined by Eq. (1), is to be applied, then it must be taken into account that its eigenvalues are complex numbers of the form $\lambda_k = a_k + i b_k$; $k = 1, 2, \dots, n$. The respective graph energy could then be calculated in two different ways:

Model 1. Following an earlier study,²⁴ the graph energy is obtained by summing the terms $|a_k|$; $k = 1, 2, \dots, n$, namely the absolute values of the real parts of the eigenvalues. The results obtained by this model are shown in Fig. 2, from which it could be seen that no correlation exists between E_π and the heats of formation of the examined polyenes. The results were by no means better by employing the alternative model.

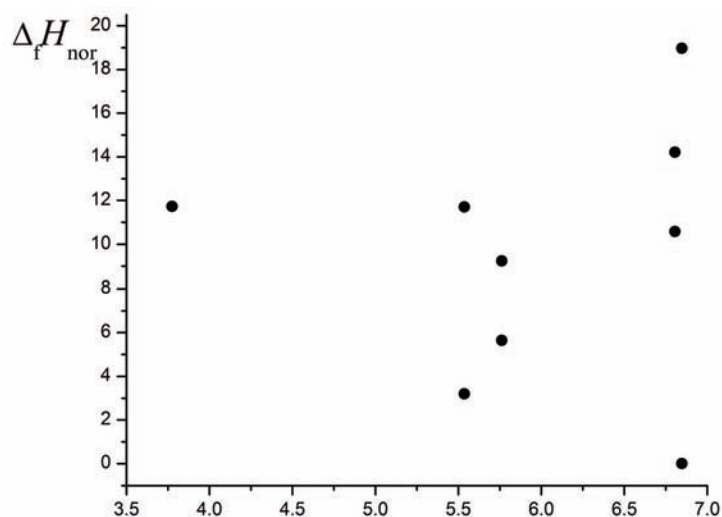


Fig. 2. Correlation between the normalized enthalpies and graph energy according to Model 1; correlation coefficient $R = 0.054$.

Model 2. The graph energy is obtained by summing the terms:

$$|a_k| + |b_k|; \quad k = 1, 2, \dots, n$$

namely, the absolute values of the real and complex parts of the eigenvalues. The obtained results are shown in Fig. 3, from which it could be seen that within Model 2 also no correlation exists between E_π and the heats of formation of the examined polyenes.

Equally disappointing results were obtained when the graph energy is the sum of terms $\sqrt{(a_k)^2 + (b_k)^2}$; $k = 1, 2, \dots, n$.

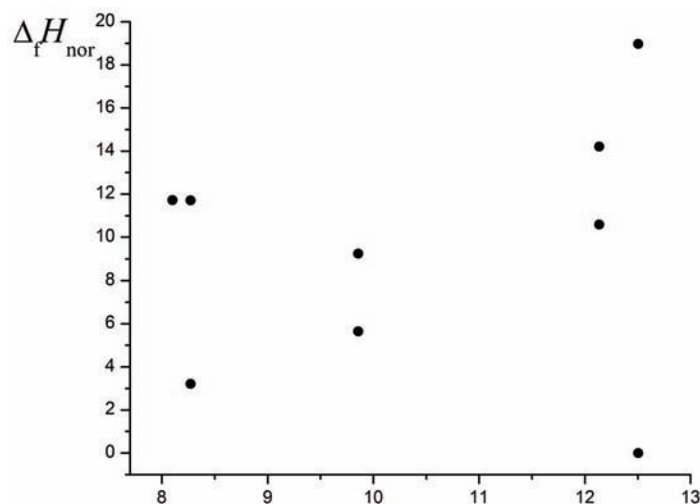


Fig. 3. Correlation between the normalized enthalpies and graph energy according to Model 2; correlation coefficient $R = 0.177$.

A THIRD APPROACH

In view of the findings outlined in the preceding section, it was concluded that in the case of an energy-based approach, Eq. (1) must be abandoned and an attempt made to construct another model, pursuing a more radical generalization.

Consider the graph $G = (V, E)$ whose vertex set is V and whose edge set is E . Suppose that there is a criterion allowing E to be decomposed into subsets E_1 and E_2 , such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$. By this, two subgraphs of G are defined, G_1 and G_2 , both with vertex sets V . G_1 will be called the principal subgraph, and G_2 the secondary subgraph. It is plausible to assume that G_1 is the ordinary molecular graph, whereas G_2 may be constructed so as to bear information on geometric (non-topological) features of the underlying molecule. The generalized adjacency matrix may then be defined as:

$$\mathbf{A}_{\text{gen}}(G) = (\mathbf{A}_1 + \mathbf{A}_2) \exp(\mathbf{A}_2) \quad (3)$$

Note that if $E_2 = \emptyset$, then $\mathbf{A}_2 = 0$ and $\mathbf{A}_{\text{gen}}(G)$ reduces to the ordinary adjacency matrix. Equation (3) can be further modified by using the Taylor series of the matrix exponent:

$$\mathbf{A}_{\text{gen}}^{(0)}(G) = \mathbf{A}_1 + \mathbf{A}_2 \quad (3a)$$

$$\mathbf{A}_{\text{gen}}^{(1)}(G) = (\mathbf{A}_1 + \mathbf{A}_2)(\mathbf{I} + \mathbf{A}_2) \quad (3b)$$

$$\mathbf{A}_{\text{gen}}^{(2)}(G) = (\mathbf{A}_1 + \mathbf{A}_2)(\mathbf{I} + \mathbf{A}_2 + \frac{1}{2}\mathbf{A}_2^2) \quad (3c)$$

and so on. Within all such decompositions, the matrices $A_{\text{gen}}^{(k)}(G)$, $k = 0, 1, 2, \dots$, are symmetric and, thus, all their eigenvalues are real-valued. Unfortunately, computations based on the higher-order members of the Taylor series did not significantly improve the model. Therefore, in the present approach, the simplest version, Eq. (3a), was used and the secondary adjacency matrix was purposefully chosen as:

$$\begin{aligned} [A_2(G)]_{ij} &= x \quad \text{if } d(v_i, v_j | G) = 3 \quad \& \quad v_i, v_j \text{ are in the } \textit{cis} \text{ constellation} \\ &= 1/x \quad \text{if } d(v_i, v_j | G) = 3 \quad \& \quad v_i, v_j \text{ are in the } \textit{trans} \text{ constellation} \quad (4) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where x is a variable parameter. This is referred to as Model 3.

Within Model 3, the energies were first computed for some chosen values of x , viz. $x = -3$ and $x = +2$. The obtained results were quite good, as can be seen from Figs. 4 and 5.

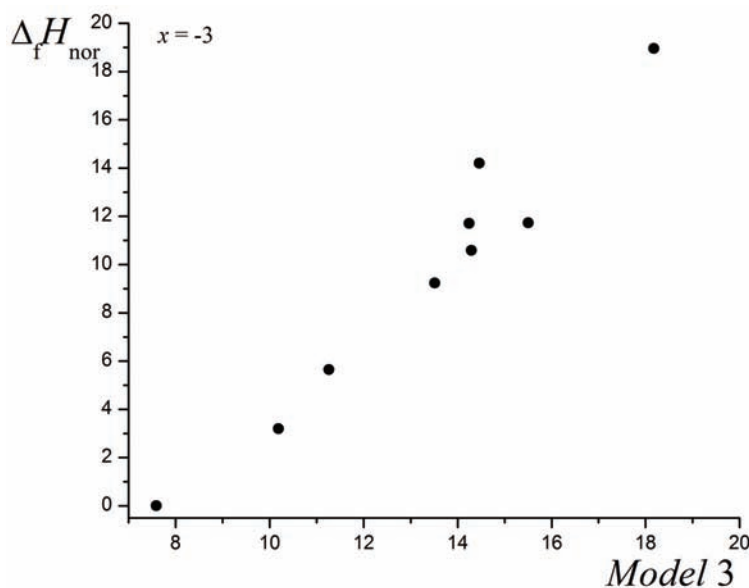


Fig. 4. Correlation between normalized enthalpies and graph energy based on the matrix $A_2(G)$ for $x = -3$; correlation coefficient $R = 0.976$.

Next, x was varied in the interval $\{-10, 10\}$ and the results shown in Fig. 6 were found. These results are surprising: except for the near-zero values of the parameter x (where instabilities and rapid changes of R occurred), the quality of the correlations were always almost the same, and were everywhere relatively good. There were insignificant maxima of the correlation coefficient at $x = -2.569$ (with a correlation coefficient $R = 0.9804$) and at $x = +2.008$ (with a

correlation coefficient $R = 0.9592$). There was also a third maximum at $x = 0.814$ (with a correlation coefficient $R = -0.9658$), but for reasons that should be evident from Fig. 6, this value of the parameter x lies in the domain of instability and is not usable for practical purposes. Thus, Model 3 yields the best results for $x = -2.569$, shown in Fig. 7.

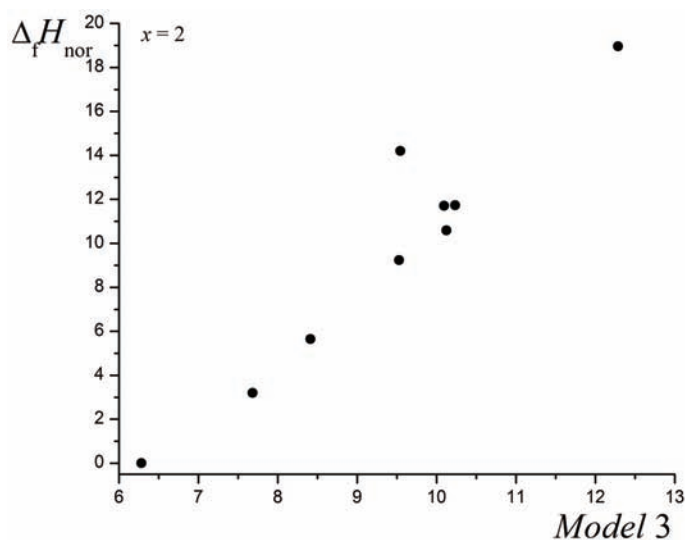


Fig. 5. Correlation between normalized enthalpies and graph energy based on the matrix $A_2(G)$ for $x = +2$; correlation coefficient $R = 0.959$.

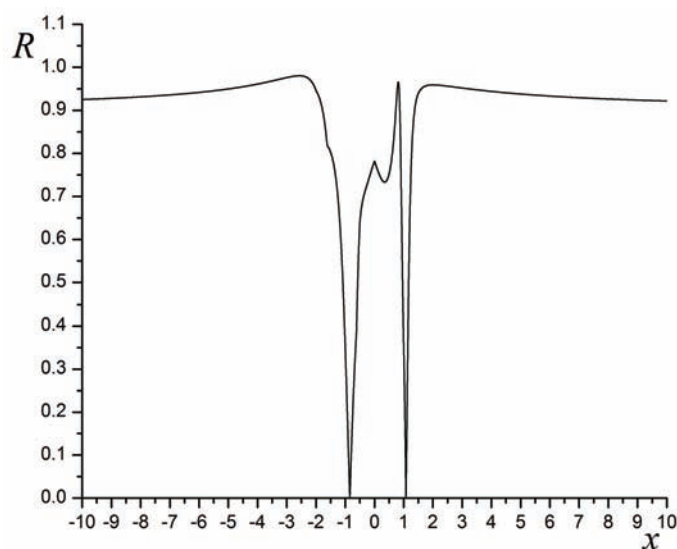


Fig. 6. Absolute value of the correlation coefficient (R) of Model 3 for the parameter x in the interval $\{-10, 10\}$.

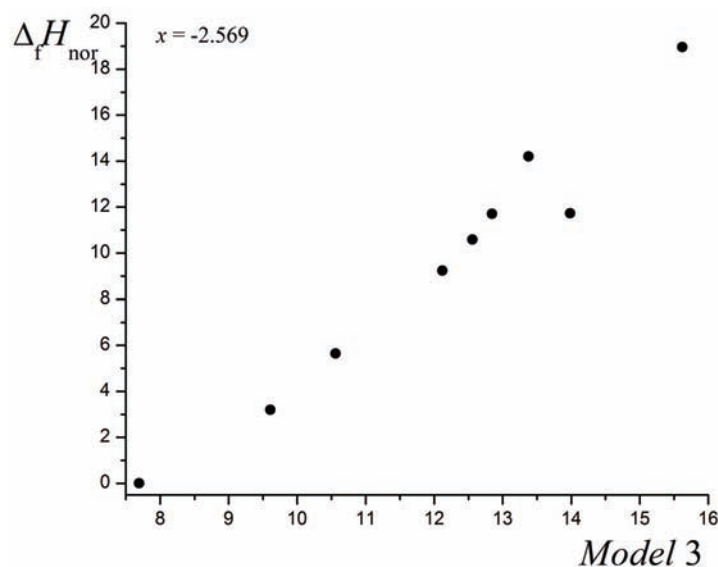


Fig. 7. The best correlation obtained within Model 3, Eqs. (3a) and (4), for $x = -2.569$; correlation coefficient $R = 0.9804$.

DISCUSSION AND CONCLUDING REMARKS

The main conclusion is that it is possible to construct a graph-based model of the *cis/trans* isomers without using complicated mathematical formalism. In particular, there was no need for complex numbers, and already the simplest modification of the adjacency matrix, namely Eqs. (3a) & (4) (= Model 3) was sufficient for “graph energy” that reasonably well reproduced the differences of thermodynamic properties (expressed *via* enthalpies) of *cis/trans* isomers. The remarkable fact that both large negative and large positive choices of the parameter x yielded agreeable results (*cf.* Fig. 6), is a further argument corroborating the proposed simple model.

Our attempts in the future will be to explore analogous graph-theoretical models for taking into account the effects of weaker molecular interactions, not necessarily based on covalent bonding. These might include intramolecular hydrogen bonding (with the secondary subgraph possessing directed edges), transition states (where the principal subgraph would consist of edges corresponding to permanent bonds whereas the secondary subgraph would represent the vanishing and emerging bonds), and van der Waals type interactions.

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ИЗВОД
ГРАНИЦЕ ЗА УКУПНУ π -ЕЛЕКТРОНСКУ ЕНЕРГИЈУ

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Предложен је једноставни граф-теоријски модел помоћу којег је могуће изразити енергетске разлике између геометријски нееквивалентних форми конјугованих полиена. То је постигнуто модификацијом матрице суседства молекулског графа, тако што је у њу укључена информација о *cis/trans* констелацијама. Укупна π -електронска енергија израчуната на овај начин је у одличној сагласности са енталпијама одговарајућих изомера и конформера.

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